ISOMETRIES OF THE UNITARY GROUPS AND THOMPSON ISOMETRIES OF THE SPACES OF INVERTIBLE POSITIVE ELEMENTS IN C^*-ALGEBRAS

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Abstract. We show that the existence of a surjective isometry (which is merely a distance preserving map) between the unitary groups of unital C^*-algebras implies the existence of a Jordan *-isomorphism between the algebras. In the case of von Neumann algebras we describe the structure of those isometries showing that any of them is extendible to a real linear Jordan *-isomorphism between the underlying algebras multiplied by a fixed unitary element. We present a result of similar spirit for the surjective Thompson isometries between the spaces of all invertible positive elements in unital C^*-algebras.

1. Introduction

The study of linear isometries between function spaces or operator algebras has a long history dating back to the early 1930's. For an excellent comprehensive treatment of related results we refer to the two volume set [5, 6]. The most fundamental and classical results of that research area are the Banach-Stone theorem describing the structure of all linear surjective isometries between the Banach spaces of continuous functions on compact Hausdorff spaces and its noncommutative generalization, Kadison's theorem [13], which describes the structure of all linear surjective isometries between general unital C^*-algebras. One immediate consequence of those results is that if two C^*-algebras are isometrically isomorphic as Banach spaces, then they are isometrically isomorphic as Jordan *-algebras, too. This provides a good example of how nicely the different sides (in the present case the linear algebraic - geometrical structure and the full algebraic, more precisely, Jordan *-algebraic structure) of one complex mathematical object may be connected to or interact with each other. We mention another famous result of similar spirit which also concerns isometries. This is the celebrated Mazur-Ulam theorem stating that any surjective isometry between normed real linear spaces is automatically an affine transformation (hence equals a real linear surjective isometry followed by a translation). This means that if two normed real linear spaces are isometric as metric spaces, then they are isometrically isomorphic as normed linear spaces, too.

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Recently, we have made attempts to extend the Mazur-Ulam theorem for more general noncommutative metrical structures, especially for metric groups. In [9] we have obtained a few general results which show that, under given conditions, the surjective isometries between certain substructures of metric groups necessarily have an algebraic property, namely they locally preserve the so-called inverted Jordan triple product $ba^{-1}b$. Applying those general results we have determined the surjective isometries of several nonlinear metrical structures (especially metric groups) of continuous functions and linear operators.

As the main motivation to our present investigations we mention that in the paper [10] we described the surjective isometries of the unitary group of a Hilbert space. Moreover, in the former paper [19] the second author determined the so-called Thompson isometries of the space of all positive definite operators on a Hilbert space. Those structures and spaces are of considerable importance due to the roles they play in several areas of algebra and analysis.

Our primary aim in this paper is to generalize the above mentioned results substantially, namely to extend them for the setting of general C^*-algebras. In what follows we show that unital C^*-algebras with isometric unitary groups are necessarily Jordan *-isomorphic. In the cases of general von Neumann algebras and commutative unital C^*-algebras we present the complete descriptions of those isometries. It turns out that they are closely related to Jordan *-isomorphisms of the underlying algebras. We obtain a similar result concerning the structure of surjective Thompson isometries between the sets of invertible positive elements of general unital C^*-algebras.

We emphasize that in this paper by an isometry we mean merely a distance preserving transformation, we do not assume that it respects an algebraic operation of any kind.

2. Norm isometries of unitary groups

In the paper [7] the first author studied surjective isometries between open subgroups of the general linear groups of unital semisimple commutative Banach algebras and proved that those transformations can uniquely be extended to isometric real linear algebra isomorphisms of the underlying algebras followed by a multiplication by a fixed element. He continued those investigations in [8] for noncommutative algebras. Using the results obtained there, in [11] the first author and K. Watanabe could give the complete description of surjective isometries between open subgroups of the general linear groups of C^*-algebras.

In what follows employing an approach very different from the one used in the above mentioned papers, we obtain structural results for the surjective isometries of certain important substructures of general linear groups. These are the unitary groups in unital C^*-algebras equipped with the usual norm and the so-called twisted subgroups of all invertible positive elements endowed with the Thompson metric. Here, by a twisted subgroup of a group G we mean a subset K of G which contains the unit of G and satisfies $yx^{-1}y \in K$ for any $x, y \in K$.

Based on our Mazur-Ulam type general results obtained in [9], we show below that formally the same description as in [11] is valid for the surjective isometries of the unitary groups of von Neumann algebras. In fact, more generally, we obtain that if the unitary groups of two unital C^*-algebras are isometric merely as metric spaces, then the underlying two algebras are (isometrically) isomorphic as Jordan *-algebras.

For our results we need the concept of Jordan isomorphisms between algebras as well as a few facts about them. If A, B are complex algebras, then a linear [real linear] map $J : A \to B$ is called a Jordan homomorphism [real linear Jordan homomorphism] if it satisfies $J(a^2) = J(a)^2$ for every $a \in A$ or, equivalently, if it satisfies $J(ab + ba) = J(a)J(b) + J(b)J(a)$ for any $a, b \in A$. Clearly, every homomorphism $\phi : A \to B$ (that is a linear map such that $\phi(ab) = \phi(a)\phi(b)$ holds for any $a, b \in A$) as well as every antihomomorphism $\psi : A \to B$ (that is a linear map $\psi : A \to B$ satisfying $\psi(ab) = \psi(b)\psi(a)$, $a, b \in A$) is a Jordan homomorphism. A Jordan *-homomorphism $J : A \to B$ between *-algebras A, B is a Jordan homomorphism which preserves the involution in the sense that $J(a^*) = J(a)^*$ holds for all $a \in A$. By a Jordan *-isomorphism we mean a bijective Jordan *-homomorphism.

In what follows the units of unital algebras will be denoted by 1. If A, B are unital algebras and $J : A \to B$ is a surjective Jordan homomorphism, then by the proof of Proposition 1.3 in [23] we have

(i) $J(1) = 1$;
(ii) $J(aba) = J(a)J(b)J(a)$, $a, b \in A$ and this implies that $J(a^n) = J(a)^n$ holds for every $a \in A$ and positive integer n;
(iii) for every invertible $a \in A$ we have that $J(a)$ is also invertible and $J(a^{-1}) = J(a)^{-1}$.

It then follows that any Jordan isomorphism between unital algebras preserves the spectrum of elements. Moreover, if A, B are unital *-algebras and $J : A \to B$ is a surjective Jordan *-homomorphism, then J maps the unitary group of A into the unitary group of B. In the case of C^*-algebras this easily implies that J is contractive due to the fact that any element of norm less than one is the arithmetic mean of unitaries (Kadison-Pedersen theorem [15]).

If A is a *-algebra, then the real linear subspace of its self-adjoint elements is denoted by A_s. If A is unital, by a symmetry in A we mean a self-adjoint unitary element (equivalently, a unitary whose square is the identity). Clearly, $s \in A$ is a symmetry if and only if it can be written as $s = 2p-1$ with a projection (self-adjoint idempotent) $p \in A$.

We are now in a position to present and prove our first result which reads as follows.

Theorem 1. *Let A_j be a unital C^*-algebra and U_j its unitary group, $j = 1, 2$. Assume $\phi : U_1 \to U_2$ is a surjective isometry (with respect to the norms given on A_1, A_2). Then we have*

$$\phi(\exp(iA_{1s})) = \phi(1)\exp(iA_{2s}) \tag{1}$$

*and there is a central projection $p \in A_2$ and a Jordan *-isomorphism $J : A_1 \to A_2$ such that*

$$\phi(\exp(ix)) = \phi(1)(pJ(\exp(ix)) + (1-p)J(\exp(ix))^*), \quad x \in A_{1s}. \tag{2}$$

Proof. Assume that $\phi : U_1 \to U_2$ is a surjective isometry. By Gelfand-Naimark theorem any C^*-algebra is isometrically *-isomorphic to a C^*-algebra of operators on a complex Hilbert space. Hence we can apply Theorem 6 in [10] stating that every surjective isometry of a subgroup of the full unitary group on a Hilbert space preserves the inverted Jordan triple product of close enough elements. Namely, for any pair $a, b \in U_1$ with $\|a - b\| < 1/2$ we have

$$\phi(ba^{-1}b) = \phi(b)\phi(a)^{-1}\phi(b).$$

(Observe that the result [10, Theorem 6] is stated only for self-maps of subgroups of the full unitary group but the same argument applies for surjective isometries between any two such subgroups acting on any two Hilbert spaces.) Pick an arbitrary self-adjoint element $x \in A_{1s}$ and consider the generated one-parameter unitary group $t \mapsto \exp(itx)$, $t \in \mathbb{R}$. Choose two arbitrary real numbers t_1 and t_2 and set $a = \exp(it_1 x), b = \exp(it_2 x)$. Select a positive integer m such that

$$\exp(\|i(t_2 - t_1)x\|/2^m) - 1 < 1/2.$$

It is apparent that

$$\|\exp(i(t_2 - t_1)x/2^m) - 1\| \leq \exp(\|i(t_2 - t_1)x\|/2^m) - 1 < 1/2$$

holds. For $k = 0, 1, 2, \ldots, 2^{m+1}$ let

$$a_k = a\exp(ik(t_2 - t_1)x/2^m).$$

Then we have $a_0 = a$, $a_{2^m} = b$, and $a_{2^{m+1}} = ba^{-1}b$. It is easy to check that $a_{k+1}a_k^{-1}a_{k+1} = a_{k+2}$ holds for every $k = 0, 1, 2, \ldots, 2^{m+1} - 2$. We also have

$$\|a_{k+1} - a_k\| = \|\exp(i(t_2 - t_1)x/2^m) - 1\| < 1/2$$

for any $k = 0, 1, 2, \ldots, 2^{m+1} - 1$. Then by the aforementioned result Theorem 6 in [10] we infer that

$$\phi(a_{k+1}a_k^{-1}a_{k+1}) = \phi(a_{k+1})\phi(a_k)^{-1}\phi(a_{k+1})$$

holds for all $k = 0, 1, 2, \ldots, 2^{m+1} - 2$. The technical Lemma 7 in [10] states that (even in a more general purely algebraic setting) the above equalities imply that the inverted Jordan triple product of a_0, a_{2^m} is also preserved, i.e., we have

$$\phi(ba^{-1}b) = \phi(a_{2^m}a_0^{-1}a_{2^m}) = \phi(a_{2^m})\phi(a_0)^{-1}\phi(a_{2^m}) = \phi(b)\phi(a)^{-1}\phi(b).$$

Therefore, we obtain that the isometry ϕ preserves the inverted Jordan triple product along any one-parameter group $(\exp(itx))_{t\in\mathbb{R}}$, $x \in A_{1s}$.

If we define

$$\phi_0(a) = \phi(1)^{-1}\phi(a), \quad a \in U_1,$$

then $\phi_0 : U_1 \to U_2$ is a surjective isometry too, which has the additional property that it sends the unit to the unit. It is then easy to verify that along one-parameter unitary groups of the previous type, ϕ_0 respecting the inverted Jordan triple product and being unital, it necessarily respects the inverse operation, i.e., we have $\phi_0(a^{-1}) = \phi_0(a)^{-1}$, and

$$\phi_0(bab) = \phi_0(b)\phi_0(a)\phi_0(b)$$

holds for every pair a, b in U_1 that belong to a one-parameter unitary group $(\exp(itx))_{t\in\mathbb{R}}$ with some $x \in A_{1s}$. One can easily deduce that

$$\phi_0(a^m) = \phi_0(a)^m$$

holds for any $a \in U_1$ of the form $a = \exp(itx)$ with some $x \in A_{1s}, t \in \mathbb{R}$ and for any integer m.

Pick $x \in A_{1s}$ and define $S_x : \mathbb{R} \to U_2$ by

$$S_x(t) = \phi_0(\exp(itx)), \quad t \in \mathbb{R}.$$

We assert that S_x is a continuous one-parameter unitary group in A_2. Since ϕ_0 is continuous, we only need to prove that $S_x(t+t') = S_x(t)S_x(t')$ holds for every pair t, t' of real numbers. First select rational numbers r and r' such that $r = \frac{n}{m}$ and $r' = \frac{n'}{m'}$ with integers m, m', n, n'. We compute

$$\begin{aligned} S_x(r+r') &= \phi_0(\exp(i\frac{nm'+mn'}{mm'}x)) = \phi_0(\exp(i\frac{1}{mm'}x))^{nm'+mn'} \\ &= \phi_0(\exp(i\frac{1}{mm'}x))^{nm'}\phi_0(\exp(i\frac{1}{mm'}))^{mn'} = S_x(r)S_x(r'). \end{aligned}$$

Since ϕ_0 is continuous, we then obtain $S_x(t+t') = S_x(t)S_x(t')$ for every pair t, t' of real numbers.

As already mentioned, we may consider A_1, A_2 as unital C^*-algebras of operators that act on the complex Hilbert spaces H_1, H_2, respectively. Applying Stone's theorem (see Section 5 in Chapter X in [2]) for the norm continuous one-parameter unitary group $(S_x(t))_{t\in\mathbb{R}}$, we infer that there exists a unique bounded self-adjoint operator y on H_2 such that $S_x(t) = \exp(ity)$ holds for every $t \in \mathbb{R}$ (the boundedness of y is the consequence of the norm continuity of S_x). Since the generator y can be obtained by differentiating $\exp(ity)$ with respect to t, where the limit of difference quotients is taken in the norm topology, it follows that $y \in A_{2s}$.

Defining $f(x) = y$ we obtain a map $f : A_{1s} \to A_{2s}$ for which

$$\phi_0(\exp(itx)) = S_x(t) = \exp(itf(x)), \quad t \in \mathbb{R}, x \in A_{1s}.$$

Since ϕ_0 is injective, f is also injective. Considering ϕ_0^{-1} in the place of ϕ_0, we infer that there is an injective map $g : A_{2s} \to A_{1s}$ such that $\phi_0^{-1}(\exp(ity)) = \exp(itg(y))$ holds for every $y \in A_{2s}$ and $t \in \mathbb{R}$. This easily implies that $y = f(g(y))$, $y \in A_{2s}$. Hence f is surjective and therefore it is a bijection from A_{1s} onto A_{2s}. In particular, it follows that the transformation ϕ_0 maps the one-parameter unitary groups $(\exp(itx))_{t\in\mathbb{R}}$, $x \in A_{1s}$ in U_1 onto the one-parameter unitary groups $(\exp(ity))_{t\in\mathbb{R}}$, $y \in A_{2s}$ in U_1. The equality (1) is now apparent.

We next prove that f is an isometry. Since ϕ_0 is an isometry, we have

$$\begin{aligned} &\| \exp(itf(x)) - \exp(itf(x'))\| \\ &\qquad = \|\phi_0(\exp(itx)) - \phi_0(\exp(itx'))\| = \| \exp(itx) - \exp(itx')\| \end{aligned}$$

for any $x, x' \in A_{1s}$ and $t \in \mathbb{R}$. Letting $t \to 0$, we have

$$\frac{\exp(itx) - \exp(itx')}{t} = \frac{\exp(itx) - 1}{t} - \frac{\exp(itx') - 1}{t} \to ix - ix'$$

and, similarly, we obtain

$$\frac{\exp(itf(x)) - \exp(itf(x'))}{t} \to if(x) - if(x').$$

It then follows that for any pair $x, x' \in A_{1s}$ we have $\|x-x'\| = \|f(x)-f(x')\|$. This gives us that f is indeed a surjective isometry. Since $f(0) = 0$, by the Mazur-Ulam theorem we infer that f is a real linear isometry from A_{1s} onto A_{2s}.

The structure of such maps was described by Kadison [14, Theorem 2]. According to his result, $f(1)$ is a central symmetry in A_2 and we have a Jordan *-isomorphism J between A_1 and A_2 such that $f(x) = f(1)J(x)$ holds for every $x \in A_{1s}$. Plainly, there is a central projection $p \in A_{2s}$ such that $f(1) = 2p - 1$ and J is isometric.

Since $(p-(1-p))^n = p+(-1)^n(1-p)$ holds for all positive integers n, we can compute

$$\phi_0(\exp(ix)) = \exp(if(x)) = \exp(if(1)J(x)) =$$
$$\exp(i(p-(1-p))J(x)) = \sum_{n=0}^{\infty} \frac{((i(p-(1-p)))J(x))^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{i^n(p-(1-p))^n J(x)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n(p+(-1)^n(1-p))J(x^n)}{n!}$$
$$= pJ\left(\sum_{n=0}^{\infty} \frac{(ix)^n}{n!}\right) + (1-p)J\left(\sum_{n=0}^{\infty} \frac{(-ix)^n}{n!}\right)$$
$$= pJ(\exp(ix)) + (1-p)J(\exp(ix))^*$$

for every $x \in A_{1s}$. Multiplying by $\phi(1)$ from the left, we obtain (2). The proof is complete. □

We remark that the idea of employing one-parameter unitary groups in the proof has been motivated by the argument in Sakai's paper [20] where he described the structure of the uniformly continuous group isomorphisms of unitary groups in AW^*-factors.

Our theorem has the following immediate corollary.

Corollary 2. *Two unital C^*-algebras are isomorphic as Jordan *-algebras if and only if their unitary groups as metric spaces are isometric.*

Proof. The necessity follows from the facts that any Jordan *-isomorphism between unital C^*-algebras maps the unitary group onto the unitary group and is isometric (see the discussion preceding the theorem above). The sufficiency is a consequence of Theorem 1. □

In the case of von Neumann algebras the structure of surjective isometries between unitary groups can be described precisely as follows.

Corollary 3. *Let M_j be a von Neumann algebra and U_j its unitary group, $j = 1, 2$. The map $\phi : U_1 \to U_2$ is a surjective isometry if and only if there is a central projection p in M_2 and a Jordan *-isomorphism $J : M_1 \to M_2$ such that ϕ is of the form*

$$\phi(a) = \phi(1)(pJ(a) + (1-p)J(a)^*), \quad a \in U_1. \tag{3}$$

Proof. It is well known that for any von Neumann algebra M its unitary group coincides with $\exp(iM_s)$. Hence the necessity part of the corollary follows from Theorem 1. As for sufficiency, if a map $\phi : U_1 \to U_2$ is of that form (3), then one can easily check that it is a surjective isometry using the properties of central projections and Jordan *-isomorphisms as in the proof of the previous corollary. □

We point out that in the previous corollary it would have been sufficient to assume that only one of the algebras M_1, M_2 is a von Neumann algebra, the other one can be assumed to be merely a unital C^*-algebra. Indeed, if say M_1 is a von Neumann algebra, M_2 is a unital C^*-algebra and ϕ is an isometry between their unitary groups, then by Theorem 1, M_1, M_2 are Jordan *-isomorphic and therefore they are isometric. By Sakai's abstract characterization of von Neumann algebras as unital C^*-algebras which, as Banach spaces, are dual spaces [21], it follows automatically that M_2 is also a von Neumann algebra.

In Theorem 1 we have described the general form of surjective isometries of unitary groups of unital C^*-algebras on the set of all exponentials of skew-symmetric elements. Namely, on that set a surjective isometry is the restriction of a real linear Jordan *-isomorphism between the underlying algebras multiplied by a fixed unitary element. The natural problem arises if the original surjective isometry between the full unitary groups can also be realized as the restriction of such a transformation. Apparently, for von Neumann algebras this is true. However, for general C^*-algebras we do not have this conclusion. As we shall see below it fails to be true even in the case of commutative C^*-algebras.

Below we describe the surjective isometries of unitary groups of the algebras of continuous functions on compact Hausdorff spaces and then present a necessary and sufficient condition in order that a given such surjective isometry can be extended to a real linear algebra *-isomorphism multiplied by a unitary element. Note that a surjective map between unital commutative C^*-algebras (or even uniform algebras) is a real linear isometry if and only if it is a real linear algebra *-isomorphism multiplied by a fixed unitary element and the general form of such transformations is known (cf. [4] and see also [18, 22]).

It is well known that every unital commutative C^*-algebra is isometrically isomorphic to the algebra of all complex-valued continuous functions on a compact Hausdorff space. In the rest of the section X and Y stand for two compact Hausdorff spaces. We denote by $C(X)$ (resp. $C_{\mathbb{R}}(X)$) the Banach algebra of all complex-valued (resp. real-valued) continuous functions on X equipped with the supremum norm $\|\cdot\|$. The unitary group of $C(X)$ (i.e., the set of functions $f \in C(X)$ with $|f| = 1$ on X) is denoted by $UC(X)$. The subgroup $\exp iC_{\mathbb{R}}(X)$ is the principal component of $UC(X)$ which can also be described as the set of all functions in $UC(X)$ that are homotopic to a constant function (we point out that those assertions can be derived, for example, from Lemma 5 below). The quotient group $UC(X)/\exp iC_{\mathbb{R}}(X)$ is usually denoted by $\pi^1(X)$ and called Bruschlinsky group. For $f \in UC(X)$, we denote the coset in $\pi^1(X)$ which contains f by $[f]$. It is known that $\pi^1(X)$ is isomorphic to the first Čech cohomology group $H^1(X)$ with integer coefficients [12].

We introduce the following notation. Given a clopen (closed and open) subset K of the compact Hausdorff space Y, for any $f \in C(Y)$ we define the function f^{*K} by

$$f^{*K}(y) = \begin{cases} f(y), & y \in K \\ \\ \overline{f(y)}, & y \in Y \setminus K. \end{cases}$$

Let us recall the following result that appeared as Corollary 4.5 in [9]. (Observe that it could be deduced from the proof of Theorem 1, too.)

Proposition 4. *Let $\phi : \exp iC_{\mathbb{R}}(X) \to \exp iC_{\mathbb{R}}(Y)$ be a surjective isometry. Then there exists a homeomorphisms Φ from Y onto X and a clopen subset K of Y such that*

$$\phi(f) = \phi(1)(f \circ \Phi)^{*K}$$

holds for all $f \in \exp iC_{\mathbb{R}}(X)$.

In the case where $\pi^1(X), \pi^1(Y)$ are both trivial, it follows that every surjective isometry $\phi : UC(X) \to UC(Y)$ extends to a real linear algebra *-isomorphism multiplied by the unitary element $\phi(1)$. However, the situation is very much different when the Bruschlinsky group is nontrivial, i.e., when $\exp iC_{\mathbb{R}}(X)$ is a proper subgroup of $UC(X)$ (as a simple example, we mention the case where X is the unit circle in the complex plane). Below we present a complete description of the surjective isometries between $UC(X)$ and $UC(Y)$ in the general case. As a consequence, we shall see that such an isometry need not be extendible to a real linear algebra *-isomorphism between $C(X)$ and $C(Y)$ multiplied by a fixed unitary element.

We begin with a lemma. The observation that a unitary element which is close enough to the identity has logarithm is well known, we present the proof in order to make our presentation more complete.

Lemma 5. *Suppose that $\pi^1(X)$ is nontrivial. Let $f \in UC(X) \setminus \exp iC_{\mathbb{R}}(X)$ and $g \in \exp iC_{\mathbb{R}}(X)$. Then $\|f - g\| = 2$.*

Proof. Suppose that $\|f - g\| \neq 2$. As f, g are unitaries, we have $\|f - g\| < 2$ and hence obtain

$$\|fg^{-1} - 1\| < 2. \tag{4}$$

We infer $(fg^{-1})(X) \subset \{z \in \mathbb{C} : |z| = 1, z \neq -1\}$. This implies that applying the principal branch Log of the logarithm, the function $\mathrm{Log}(fg^{-1})$ is a well defined continuous function on X which can be written as a real-valued function multiplied by the imaginary unit i. We know that there is a function $h \in C_{\mathbb{R}}(X)$ such that $g = \exp ih$. Therefore,

$$f = \exp(\mathrm{Log}(fg^{-1})) \exp ih = \exp(\mathrm{Log}(fg^{-1}) + ih) \in \exp iC_{\mathbb{R}}(X)$$

which is a contradiction and the proof of the lemma is complete. $\square$

The following auxiliary result will be used several times in the rest of the section.

Lemma 6. *Let Φ and Φ' be homeomorphisms from Y onto X, and let K and K' be clopen subsets of Y. Assume that $g, h \in UC(Y)$, $[\alpha] \in \pi^1(X)$ and*

$$g(f \circ \Phi)^{*K} = h(f \circ \Phi')^{*K'}, \quad f \in [\alpha]. \tag{5}$$

Then we have $K = K'$, $\Phi = \Phi'$, and $g = h$.

Proof. For any $f \in [\alpha]$ we have $if \in [\alpha]$. Inserting these two functions into (5) we easily obtain $i^{*K} = i^{*K'}$ implying $K = K'$. To see that $\Phi = \Phi'$ holds true, assume on the contrary that there is $y \in Y$ such that $\Phi(y) \neq \Phi'(y)$. For arbitrary complex numbers z, z' of modulus 1, by Urysohn lemma we can choose a function in $\exp iC_{\mathbb{R}}(X)$ which takes the value z at $\Phi(y)$ and takes the value z' at $\Phi'(y)$. Since for any particular $f_0 \in [\alpha]$ we have $[\alpha] = f_0 \exp iC_{\mathbb{R}}(X)$, we could deduce that there is an element $f \in [\alpha]$ for which the equality (5) does not hold at the point y which is a contradiction. Finally, the last assertion that $g = h$ is trivial. $\square$

The next theorem gives a complete description of surjective isometries between unitary groups of commutative C^*-algebras.

Theorem 7. *The map $\phi : UC(X) \to UC(Y)$ is a surjective isometry if and only if the following hold. First, the map from $\pi^1(X)$ to $\pi^1(Y)$ defined by $[f] \mapsto [\phi(f)]$ $(f \in UC(X))$ is well defined and bijective. Second, there exists a map u from $\pi^1(X)$*

into $UC(Y)$, and for every $[\alpha] \in \pi^1(X)$ there exists a homeomorphism $\Phi_{[\alpha]} : Y \to X$ and a clopen subset $K_{[\alpha]}$ of Y such that

$$\phi(f) = u([\alpha])(f \circ \Phi_{[\alpha]})^{*K_{[\alpha]}}, \quad f \in [\alpha]. \tag{6}$$

Proof. Assume first that ϕ is a surjective isometry from $UC(X)$ onto $UC(Y)$. Since, by Lemma 5, $\exp iC_{\mathbb{R}}(X)$ is closed open and connected, $UC(X)$ writes as the disjoint union of the connected sets of the form $f \exp iC_{\mathbb{R}}(X)$, where each f is taken in a different coset in $\pi^1(X)$, and so, the connected components of $UC(X)$ are exactly the cosets in $\pi^1(X)$. Therefore, the map $[f] \mapsto [\phi(f)]$, $f \in UC(X)$ is a well defined bijective transformation from $\pi^1(X)$ onto $\pi^1(Y)$.

For a given $[\alpha] \in \pi^1(X)$ pick an element $f_{[\alpha]} \in [\alpha]$. Define the auxiliary map $\psi : \exp iC_{\mathbb{R}}(X) \to \exp iC_{\mathbb{R}}(Y)$ by $\psi(f) = \phi(f_{[\alpha]})^{-1}\phi(f_{[\alpha]}f)$, $f \in \exp iC_{\mathbb{R}}(X)$. Clearly, ψ is a well defined surjective isometry from $\exp iC_{\mathbb{R}}(X)$ onto $\exp iC_{\mathbb{R}}(Y)$ with $\psi(1) = 1$. By Proposition 4 there exists a homeomorphism $\Phi_{[\alpha]} : Y \to X$ and a clopen subset $K_{[\alpha]}$ of Y such that

$$\psi(g) = (g \circ \Phi_{[\alpha]})^{*K_{[\alpha]}}, \quad g \in \exp iC_{\mathbb{R}}(X).$$

We then obtain

$$\phi(f) = \phi(f_{[\alpha]})((f_{[\alpha]}^{-1} f) \circ \Phi_{[\alpha]})^{*K_{[\alpha]}}, \quad f \in [\alpha]. \tag{7}$$

We show that here $\Phi_{[\alpha]}$ and $K_{[\alpha]}$ are independent of the choice of $f_{[\alpha]}$ in $[\alpha]$. Let $f'_{[\alpha]} \in [\alpha]$ and assume

$$\phi(f) = \phi(f'_{[\alpha]})(((f'_{[\alpha]})^{-1} f) \circ \Phi'_{[\alpha]})^{*K'_{[\alpha]}}, \quad f \in [\alpha]. \tag{8}$$

holds for a homeomorphism $\Phi'_{[\alpha]} : Y \to X$ and a clopen subset $K'_{[\alpha]}$ of Y. We then have

$$\begin{aligned} &\phi(f_{[\alpha]})(f_{[\alpha]}^{-1} \circ \Phi_{[\alpha]})^{*K_{[\alpha]}} \cdot (f \circ \Phi_{[\alpha]})^{*K_{[\alpha]}} \\ &\qquad = \phi(f'_{[\alpha]})((f'_{[\alpha]})^{-1} \circ \Phi'_{[\alpha]})^{*K'_{[\alpha]}} \cdot (f \circ \Phi'_{[\alpha]})^{*K'_{[\alpha]}}, \quad f \in [\alpha]. \end{aligned}$$

Applying Lemma 6 we obtain $K_{[\alpha]} = K'_{[\alpha]}$, $\Phi_{[\alpha]} = \Phi'_{[\alpha]}$ and

$$\phi(f_{[\alpha]})(f_{[\alpha]}^{-1} \circ \Phi_{[\alpha]})^{*K_{[\alpha]}} = \phi(f'_{[\alpha]})((f'_{[\alpha]})^{-1} \circ \Phi'_{[\alpha]})^{*K'_{[\alpha]}}.$$

This latter equality shows that

$$\phi(f_{[\alpha]})(f_{[\alpha]}^{-1} \circ \Phi_{[\alpha]})^{*K_{[\alpha]}}$$

is independent of the choice of $f_{[\alpha]}$ in $[\alpha]$. For $[\alpha] \in \pi^1(X)$ choose any $f_{[\alpha]} \in [\alpha]$. Put $u([\alpha]) = \phi(f_{[\alpha]})(f_{[\alpha]}^{-1} \circ \Phi_{[\alpha]})^{*K_{[\alpha]}}$. Then the map u from $\pi^1(X)$ into $UC(Y)$ is well defined and by (7) we get the formula

$$\phi(f) = u([\alpha])(f \circ \Phi_{[\alpha]})^{*K_{[\alpha]}}, \quad f \in [\alpha].$$

This was the necessity part of the theorem. Using the fact that the distance between the elements of $UC(X)$ belonging to different cosets are always the same (namely, 2) which follows from Lemma 5, the sufficiency part should be easy to check. □

Observe that by Lemma 6, the homeomorphisms $\Phi_{[\alpha]} : Y \to X$ and the clopen sets $K_{[\alpha]}$, $[\alpha] \in \pi^1(X)$ in (6) are uniquely determined.

We now give a necessary and sufficient condition for a surjective isometry $\phi : UC(X) \to UC(Y)$ to be extendible to a real linear isometry from $C(X)$ onto $C(Y)$.

This will clearly show that in general we do not have that extendibility property of the isometries of unitary groups of commutative C^*-algebras.

Corollary 8. *Let $\phi : UC(X) \to UC(Y)$ be a surjective isometry. Consider the representation of ϕ given in Theorem 7, i.e., for every $[\alpha] \in \pi^1(X)$ take the clopen subset $K_{[\alpha]}$ of Y and the homeomorphism $\Phi_{[\alpha]} : Y \to X$ such that*

$$\phi(f) = u([\alpha])(f \circ \Phi_{[\alpha]})^{*K_{[\alpha]}}, \quad f \in [\alpha].$$

*The map ϕ can be extended to a real linear algebra *-isomorphism from $C(X)$ onto $C(Y)$ multiplied by the unitary element $\phi(1)$ if and only if all $\Phi_{[\alpha]}$'s as well as all $K_{[\alpha]}$'s coincide and u is a constant map with the value $\phi(1)$. Moreover, in this latter case, denoting $\Phi = \Phi_{[\alpha]}$ and $K = K_{[\alpha]}$, $[\alpha] \in \pi^1(X)$, the transformation*

$$\widetilde{\phi}(f) = \phi(1)(f \circ \Phi)^{*K}, \quad f \in C(X)$$

*which is a real linear algebra *-isomorphism multiplied by $\phi(1)$ extends ϕ.*

Proof. Suppose that ϕ can be extended to $\widetilde{\phi}$, a real linear algebra *-isomorphism from $C(X)$ onto $C(Y)$ multiplied by $\phi(1)$. The structure of all real linear (or even additive) algebra *-isomorphisms between $C(X)$ and $C(Y)$ is known, see, e.g., [22, Theorem 5.2] (cf. [18, Theorem 1.1]). Using that result we obtain that there exists a homeomorphism $\Phi : Y \to X$ and a clopen subset K of Y such that

$$\widetilde{\phi}(f) = \phi(1)(f \circ \Phi)^{*K}, \quad f \in C(X).$$

On the other hand, by Theorem 7 we have that

$$\phi(f) = u([\alpha])(f \circ \Phi_{[\alpha]})^{*K_{[\alpha]}}, \quad f \in [\alpha].$$

Since $\widetilde{\phi}$ is an extension of ϕ, for every $[\alpha] \in \pi^1(X)$ we have

$$u([\alpha])(f \circ \Phi_{[\alpha]})^{*K_{[\alpha]}} = \phi(1)(f \circ \Phi)^{*K}, \quad f \in [\alpha].$$

Using Lemma 6 we obtain that $K_{[\alpha]} = K$, $\Phi_{[\alpha]} = \Phi$, and $\phi(1) = u([\alpha])$ for every $[\alpha] \in \pi^1(X)$.

Conversely, assume that $\Phi_{[\alpha]} = \Phi$, $K_{[\alpha]} = K$, and $u([\alpha]) = \phi(1)$ hold for every $[\alpha] \in \pi^1(X)$. The map $f \mapsto (f \circ \Phi)^{*K}$ is clearly a real linear algebra *-isomorphism from $C(X)$ onto $C(Y)$ and by Theorem 7 the transformation $\widetilde{\phi}$ defined by

$$\widetilde{\phi}(f) = \phi(1)(f \circ \Phi)^{*K}, \quad f \in C(X)$$

extends ϕ. This completes the proof. □

3. Thompson isometries of the spaces of invertible positive elements in C^*-algebras

In this section we determine the Thompson isometries of the spaces of invertible positive elements of unital C^*-algebras. The Thompson metric (or Thompson part metric) can be defined in a rather general setting involving normed linear spaces and certain closed cones, see [24]. This metric has a wide range of applications from non-linear integral equations, linear operator equations, ordinary differential equations to optimal filtering and beyond. Given a unital C^*-algebra A, the general definition of the Thompson metric d_T on the set A_+^{-1} of its invertible positive elements reads

$$d_T(a,b) = \log\max\{M(a/b), M(b/a)\}, \quad a, b \in A_+^{-1}, \tag{9}$$

where $M(x/y) = \inf\{t > 0 : x \le ty\}$ for any $x, y \in A_+^{-1}$. It is easy to see that it can be rewritten as

$$d_T(a,b) = \left\|\log\left(a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)\right\|, \quad a,b \in A_+^{-1}$$

(see, e.g., [19]).

In the setting of C^*-algebras, the Thompson metric has important differential geometrical connections. To see this, observe that A_+^{-1} is an open subset of the Banach space A_s of all self-adjoint elements of A and hence it is a differentiable manifold which in fact carries a natural Finsler geometrical structure (for more details and for further reading, see e.g., [1]). At any point $a \in A_+^{-1}$, the tangent space is identified with the linear space A_s in which the norm of a vector x is defined as $\|a^{-\frac{1}{2}}xa^{-\frac{1}{2}}\|$. It turns out that in the so-obtained Finsler space the geodesic distance d can be computed as

$$d(a,b) = \left\|\log\left(a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)\right\|, \quad a,b \in A_+^{-1}$$

which coincides with the Thompson metric d_T on A_+^{-1}.

We point out that the differential geometry of A_+^{-1} is an active research area with many applications. Indeed, even what represents the finite dimensional case, the differential geometry of the space of $n \times n$ positive definite matrices has important applications among others in linear systems, statistics, filters, Lagrangian geometry and quantum systems (see, e.g., [3]).

As we have already mentioned above, in the paper [19] the second author determined the Thompson isometries of A_+^{-1} in the particular case of the full operator algebra $A = B(H)$ of all bounded linear operators acting on the complex Hilbert space H. The following theorem gives a far reaching generalization of that result by describing the structure of Thompson isometries in the setting of general unital C^*-algebras. In particular, the result says that if the spaces of all invertible positive elements of unital C^*-algebras are isometric with respect to the Thompson metric (again, merely as metric spaces), then the underlying algebras are isomorphic as Jordan *-algebras. Hence, one can say that the metrical - differential geometrical structure of the space of invertible positive elements completely determines the Jordan algebraic structure of the underlying C^*-algebra. The result may be viewed as a differential geometry related counterpart of Kadison's famous theorem mentioned in the introduction.

Before presenting the precise statement and its proof we recall an important correspondence between the spectra of ab and ba, where a, b are elements of a unital complex algebra A. Namely, we have $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$. In particular, if a, b are invertible, it follows that $\sigma(ab) = \sigma(ba)$.

Theorem 9. *Let A_j be a unital C^*-algebra and A_{j+}^{-1} the set of all invertible positive elements in A_j, $j = 1, 2$. The map $\phi : A_{1+}^{-1} \to A_{2+}^{-1}$ is a surjective isometry with respect to the Thompson metric if and only if there is a central projection p in A_2 and a Jordan *-isomorphism $J : A_1 \to A_2$ such that ϕ is of the form*

$$\phi(a) = \phi(1)^{1/2}\left(pJ(a) + (1-p)J(a^{-1})\right)\phi(1)^{1/2}, \quad a \in A_{1+}^{-1}. \tag{10}$$

Proof. Let us begin with the sufficiency part. Assume that ϕ is of the form (10) above. Observe that, by the spectral mapping theorem, the Thompson metric is a function of the spectra

$$\sigma\left(a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right) = \sigma\left(a^{-1}b\right), \quad a, b \in A_{1+}^{-1},$$

namely the maximal modulus of the logarithm of their elements. Since a Jordan *-isomorphism preserves the Jordan triple product aba, positivity, the square roots of positive elements, the inverse, and the spectra of elements, it then follows that it preserves the Thompson distance as well. Considering the above displayed spectra, in particular, the one on the left hand side, it is easy to verify that the inverse operation $a \mapsto a^{-1}$ is also a Thompson isometry on A_{1+}^{-1}. Moreover, considering the spectra on the right hand side, one can shortly check that for any invertible $t \in A_1$, the transformation $a \mapsto tat^*$ is a Thompson isometry, too. It is now apparent that any map ϕ of the form (10) with some central projection $p \in A_2$ and Jordan *-isomorphism $J : A_1 \to A_2$ is a surjective Thompson isometry.

As for the necessity, assume that ϕ is a Thompson isometry. Set $\phi_0(.) = \phi(1)^{-1/2}\phi(.)\phi(1)^{-1/2}$. It follows from the above discussion that ϕ_0 is a Thompson isometry which maps the identity to the identity. It was proven in [10, Theorem 9] that the surjective Thompson isometries between A_{1+}^{-1} and A_{2+}^{-1} globally preserve the inverted Jordan triple product. As ϕ_0 is unital too, it follows that

$$\phi_0(aba) = \phi_0(a)\phi_0(b)\phi_0(a)$$

holds for every pair a, b in A_{1+}^{-1}. Inserting $b = 1$ into this equality we deduce that $\phi_0(a^2) = \phi_0(a)^2$ implying that $\phi_0(a^{1/2}) = \phi_0(a)^{1/2}$. One can further show that

$$\phi_0(a^{1/n}) = \phi_0(a)^{1/n} \tag{11}$$

holds for any $a \in A_{1+}^{-1}$ and positive integer n.

Let us now consider the bijective transformation S from A_{1s} onto A_{2s} defined by $S(x) = \log \phi_0(\exp x)$. For every positive integer n we obtain by (11) that

$$S\left(\frac{x}{n}\right) = \log \phi_0\left(\exp\left(\frac{x}{n}\right)\right) = \frac{S(x)}{n} \tag{12}$$

holds for every $x \in A_{1s}$. Using the formula

$$d_T(a, b) = \left\|\log\left(a^{-1/2}ba^{-1/2}\right)\right\|, \quad a, b \in A_+^{-1}$$

of the Thompson metric we have

$$d_T\left(\exp\left(\frac{x}{n}\right), \exp\left(\frac{y}{n}\right)\right) = \left\|\log\left(\exp\left(-\frac{x}{2n}\right)\exp\left(\frac{y}{n}\right)\exp\left(-\frac{x}{2n}\right)\right)\right\|.$$

We can write

$$\begin{aligned} &\exp\left(-\frac{x}{2n}\right)\exp\left(\frac{y}{n}\right)\exp\left(-\frac{x}{2n}\right) \\ &= 1 + \frac{1}{n}\left(-\frac{x}{2} + y - \frac{x}{2}\right) + o\left(\frac{1}{n}\right) = 1 + \frac{1}{n}(y - x) + o\left(\frac{1}{n}\right), \end{aligned}$$

where the term $o(\frac{1}{n})$ belongs to A_{1s} and $o(\frac{1}{n})n \to 0$ as $n \to \infty$. Hence

$$\frac{d_T(\exp(\frac{x}{n}), \exp(\frac{y}{n}))}{1/n} = \left\|\frac{\log\left(1 + \frac{1}{n}(y - x) + o(\frac{1}{n})\right)}{1/n}\right\| \to \|y - x\|$$

as $n \to \infty$. In fact, to see this convergence one can apply the power series expansion $\log(1 + a) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}a^n}{n}$ of the logarithmic function which holds in any unital

Banach algebra for any element a with norm less than 1. Obviously, by (12) we similarly have

$$\frac{d_T(\phi_0(\exp(\frac{x}{n})), \phi_0(\exp(\frac{y}{n})))}{1/n} = \left\| \frac{d_T(\exp\left(\frac{S(x)}{n}\right), \exp\left(\frac{S(y)}{n}\right))}{1/n} \right\| \to \|S(y) - S(x)\|$$

as $n \to \infty$. Since ϕ_0 is a Thompson isometry, we have $d_T(\exp(\frac{x}{n}), \exp(\frac{y}{n})) = d_T(\phi_0(\exp(\frac{x}{n})), \phi_0(\exp(\frac{y}{n})))$ for every n. This implies that $\|x - y\| = \|S(x) - S(y)\|$, i.e., $S : A_s \to B_s$ is an isometry. It is clear that S is bijective and $S(0) = 0$. By the Mazur-Ulam theorem we conclude that S is a real linear isometry from A_{1s} onto A_{2s}.

Similarly to the proof of Theorem 1, we can now apply the result [14, Theorem 2] of Kadison. We obtain that there is a central projection $p \in A_{2s}$ and a Jordan *-isomorphism $J : A_1 \to A_2$ such that $S(1) = 2p - 1$ and $S(x) = S(1)J(x)$ holds for every $x \in A_{1s}$. We next compute

$$\begin{aligned}
\phi_0(\exp x) = \exp S(x) = \exp((p - (1-p))J(x)) &= \sum_{n=0}^{\infty} \frac{((p-(1-p))J(x))^n}{n!} \\
= \sum_{n=0}^{\infty} \frac{(p-(1-p))^n J(x)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(p + (-1)^n(1-p))J(x^n)}{n!} \\
&= pJ\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) + (1-p)J\left(\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) \\
&= pJ(\exp x) + (1-p)J(\exp(-x))
\end{aligned}$$

for every $x \in A_{1s}$. Consequently, we have

$$\phi(a) = \phi(1)^{\frac{1}{2}}\phi_0(a)\phi(1)^{\frac{1}{2}} = \phi(1)^{\frac{1}{2}}(pJ(a) + (1-p)J(a^{-1}))\phi(1)^{\frac{1}{2}}, \quad a \in A_{1+}^{-1}.$$

The proof of the theorem is complete. □

We make the following hopefully interesting observation. As pointed out in [1], for a unital C^*-algebra A the unique geodesic between the elements a, b in the Finsler space A_+^{-1} is given by

$$\gamma_{a,b}(t) = a^{\frac{1}{2}}\left(a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^t a^{\frac{1}{2}}, \quad t \in [0,1].$$

The right hand side of this formula is in connection with an important concept in operator theory. Let H be a complex Hilbert space. For an arbitrary real number $0 \leq t \leq 1$, the formula

$$a \#_t b = a^{\frac{1}{2}}\left(a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^t a^{\frac{1}{2}}$$

defines an operator mean for the invertible positive operators $a, b \in B(H)$ in the sense of Kubo and Ando [16]. By the transfer property (again, see [16]) we have

$$x(a\#_t b)x^* = (xax^*)\#_t(xbx^*)$$

for any invertible operator $x \in B(H)$. Since Jordan *-isomorphisms are easily seen to preserve any real powers of invertible positive elements (indeed, they preserve positive integer powers and inverses, hence all integer powers of invertible elements too; they also preserve positive elements and hence positive integer roots of positive elements; finally, by continuity, they preserve arbitrary real powers of invertible positive elements), it then follows from the theorem above that for any Thompson isometry $\phi : A_{1+}^{-1} \to A_{2+}^{-1}$ we necessarily have

$$\phi(\gamma_{a,b}(t)) = \gamma_{\phi(a),\phi(b)}(t), \quad t \in [0,1].$$

This shows the interesting fact that the geodesic distance preserving maps also preserve the geodesic curves themselves in the above sense.

Finally, as another remark we mention that one may naturally ask for the structure of the usual norm isometries between A_{1+}^{-1} and A_{2+}^{-1}. The fact is that this problem is not so exciting, it can be solved easily by applying some known results. Indeed, by a theorem of Mankiewicz [17] every surjective isometry from an open connected subset of a normed space E onto an open connected subset of another normed space F can be extended to an affine isometry from E onto F. Clearly, the sets A_{1+}^{-1} and A_{2+}^{-1} are open convex subsets in A_{1s} and A_{2s}, respectively. Therefore, if $\phi : A_{1+}^{-1} \to A_{2+}^{-1}$ is a surjective isometry with respect to the usual norm, then it can be extended to a surjective affine isometry $\tilde{\phi} : A_{1s} \to A_{2s}$. Clearly, by continuity $\tilde{\phi}$ maps the positive cone A_{1+} (the set of all positive elements of A_1) onto the positive cone A_{2+} and, by affinity, it sends the unique extremal point 0 of A_{1+} to the unique extremal point 0 of A_{2+}. It follows that $\tilde{\phi}$ is in fact a surjective linear isometry and then the result [14, Theorem 2] can again be applied to verify that ϕ can be extended to a Jordan *-isomorphism $J : A_1 \to A_2$.

References

[1] E. Andruchow, G. Corach and D. Stojanoff, *Geometrical significance of the Löwner-Heinz inequality,* Proc. Amer. Math. Soc. **128** (2000), 1031–1037.

[2] J. B. Conway, *A Course in Functional Analysis,* Springer-Verlag, 1990.

[3] G. Corach, A. Maestripieri and D. Stojanoff, *Orbits of positive operators from a differentiable viewpoint,* Positivity **8** (2004), 31–48.

[4] A. J. Ellis, *Real characterizations of function algebras amongst function spaces,* Bull. London Math. Soc., **22** (1990), 381–385.

[5] R. J. Fleming and J. E. Jamison, *Isometries on Banach Spaces: Function Spaces,* Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics. 129, Boca Raton, 2003.

[6] R. J. Fleming and J. E. Jamison, *Isometries in Banach Spaces: Vector-valued Function Spaces and Operator Spaces,* Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics. 138, Boca Raton, 2007.

[7] O. Hatori, *Isometries between groups of invertible elements in Banach algebras,* Studia Math., **194** (2009), 293–304.

[8] O. Hatori, *Algebraic properties of isometries between groups of invertible elements in Banach algebras,* J. Math. Anal. Appl. **376** (2011), 84–93.

[9] O. Hatori, G. Hirasawa, T. Miura and L. Molnár, *Isometries and maps compatible with inverted Jordan triple products on groups,* Tokyo J. Math. **35** (2012), 385–410.

[10] O. Hatori and L. Molnár, *Isometries of the unitary group,* Proc. Amer. Math. Soc. **140** (2012), 2141–2154.

[11] O. Hatori and K. Watanabe, *Isometries between groups of invertible elements in C^*-algebras,* Studia Math. **209** (2012), 103–106.

[12] S. T. Hu, *Homotopy Theory,* Academic Press, New York, 1959.

[13] R. V. Kadison, *Isometries of operator algebras,* Ann. Math. **54** (1951), 325–338.

[14] R. V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algebras,* Ann. Math. **56** (1952), 494–503.

[15] R. V. Kadison and G. K. Pedersen, *Means and convex combinations of unitary operators,* Math. Scand. **57** (1985), 249–266.

[16] F. Kubo and T. Ando, *Means of positive linear operators,* Math. Ann. **246** (1980), 205–224.

[17] P. Mankiewicz, *On extension of isometries in normed linear spaces,* Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **20** (1972), 367–371.
[18] T. Miura, *Real-linear isometries between function algebras,* Cent. Eur. J. Math. **9** (2011), 778–788.
[19] L. Molnár, *Thompson isometries of the space of invertible positive operators,* Proc. Amer. Math. Soc., **137** (2009), 3849–3859.
[20] S. Sakai, *On the group isomorphism of unitary groups in* AW^**-algebras,* Tohoku Math. J., **7** (1955), 87–95.
[21] S. Sakai, *A characterization of* W^**-algebras,* Pacific J. Math. **6** (1956), 763–773.
[22] P. Šemrl, *Non-linear perturbations of homomorphisms on* $C(X)$*,* Quart. J. Math. Oxford **50** (1999), 87–109.
[23] A. R. Sourour, *Invertibility preserving linear maps on* $L(X)$*,* Trans. Amer. Math. Soc. **348** (1996) 13–30.
[24] A. C. Thompson, *On certain contraction mappings in a partially ordered vector space,* Proc. Amer. Math. Soc. **14** (1963), 438–443.

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